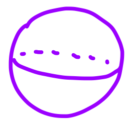
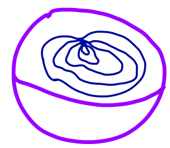
# I. Fundamental Group and Covering Spaces

## A. Fundamental Group

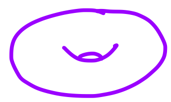
the basic idea is to "probe the topology of a space with loops mapped into the space"

intuitively any loop in  $S^2$ can be "shrunk to a point"

i.e. homotoped to a constant loop

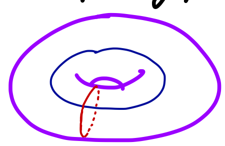


but there are loops in

$T^2 =$ 

that get "caught on the topology" and cannot be shrunk.

find "holes" in the space



rigorously, as we said above the fundamental group of a topological space $X$ with a base point $x_0 \in X$ is

$$\pi_1(X, x_0) = [S^1, X]_0$$

homotopy classes of based maps from $S^1$ to $X$

we want to see a group structure on this, to this end we need

**exercise:** let $S^1 \subset \mathbb{R}^2$ be the unit circle

$$p : [0,1] \to S^1$$
$$t \mapsto (\cos 2\pi t, \sin 2\pi t)$$

is a quotient map (i.e. can think of $S^1$ as $[0,1]$ with end pts identified)

moreover, there is a one-to-one correspondence

between $\gamma : ([0,1], \{0,1\}) \to (X, x_0)$ call this a based loop in $X$

and $\hat{\gamma} : (S^1, \{(1,0)\}) \to (X, x_0)$

( given by $\hat{\gamma} \circ p = \gamma$ )

so $[S^1, X]_0$ is the same as $[([0,1],\{0,1\}), (X, x_0)]$ homotopy, rel end pts, classes of loops in $X$ based at $x_0$

if $\gamma : [0,1] \to X$ a based loop, then its homotopy class is denoted $[\gamma]$

if $\gamma_1, \gamma_2$ are two loops based at $x_0$ then define

$\gamma_1 * \gamma_2$ to be the loop

$$\gamma_1 * \gamma_2 : [0,1] \to X : t \longmapsto \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ \gamma(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

i.e. go around $\gamma_1$ then around $\gamma_2$

$\gamma_1 * \gamma_2$ is clearly well-defined on loops, but is it well-defined on homotopy classes of loops?

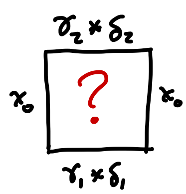
let $\gamma_1 \sim \gamma_2$ by homotopy $H : [0,1] \times [0,1] \to X$

$\delta_1 \sim \delta_2$ " " $G : [0,1] \times [0,1] \to X$

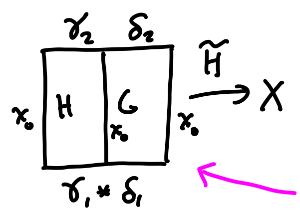
we need to find a homotopy $\gamma_1 * \delta_1$ to $\gamma_2 * \delta_2$

that is a map $\tilde{H} : [0,1] \times [0,1] \to X$ s.t.

$$\tilde{H}(t,0) = \gamma_1 * \delta_1$$
$$\tilde{H}(t,1) = \gamma_2 * \delta_2$$
$$\tilde{H}(i,s) = x_0 \quad i = 0,1, \ \forall s$$

i.e. 

$\longrightarrow X$

we can fill in ? with $H$ and $G$: 


can use such pictures to get idea for homotopy

rigorously $$\tilde{H}(t,s) = \begin{cases} H(2t,s) & 0 \le t \le \frac{1}{2} \\ G(2t-1,s) & \frac{1}{2} \le t \le 1 \end{cases}$$

so $[\gamma_1] * [\delta_1] = [\gamma_1 * \delta_1]$ is well-defined!

<u>lemma 1:</u>

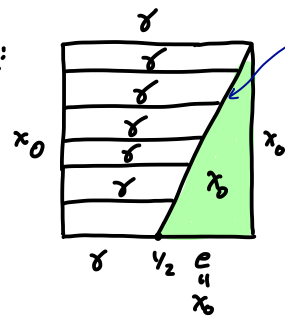
$\boxed{(\pi_1(X, x_0), *) \text{ is a group}}$

**Proof:**

**identity:** let $e:[0,1]\to X : t \mapsto x_0$ constant loop

**note:** $[e]*[\gamma]=[\gamma]=[\gamma]*[e]$

**Picture:**

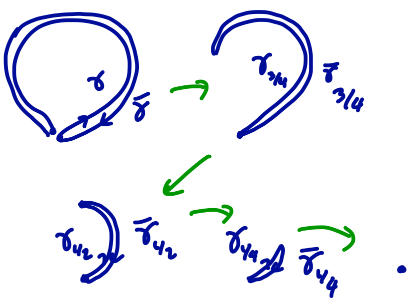



this line is $s=2t-1$, $t=\frac{s+1}{2}$

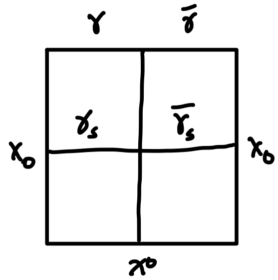
**translate:**

$$H(t,s)=\begin{cases}\gamma\left(\frac{2t}{1+s}\right) & 0\le t\le \frac{1+s}{2}\\ x_0 & \frac{1+s}{2}\le t\le 0\end{cases}$$

**inverses:** given $[\gamma]$, then $[\gamma]^{-1}=[\bar\gamma]$ where $\bar\gamma(t)=\gamma(1-t)$

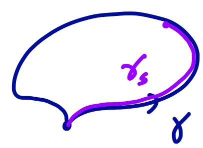



**Picture:**




where $\gamma_s(t)=\gamma(st)$

**note** $\bar\gamma_s(t)=\gamma(s-st)$

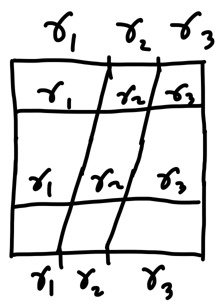



rigorously

$$H(t,s)=\begin{cases}\gamma_s(2t) & t\le 1/2\\ \bar\gamma_s(2t-1) & t\ge 1/2\end{cases}=\begin{cases}\gamma(2st) & t\le 1/2\\ \gamma(s-s(2t-1)) & t\ge 1/2\end{cases}$$

**associativity:** need to see $(\gamma_1*\gamma_2)*\gamma_3\sim\gamma_1*(\gamma_2*\gamma_3)$

**Picture:**




**exercise:** write out $H$.

If $f:X\to Y$ a map

$x_0\in X$ any $y_0=f(x_0)$

then given any loop $\gamma:[0,1]\to X$ based at $x_0$

we get a loop $f\circ\gamma:[0,1]\to Y$ based at $y_0$

**exercise:** If $\gamma\sim\delta$ then $f\circ\gamma\sim f\circ\delta$

so $f$ induces a map

$$f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$$
$$[\gamma] \longmapsto [f \circ \gamma]$$

**lemma 2:**

$$\boxed{f_* \text{ is a homomorphism}}$$

**Proof:** $[\gamma_1], [\gamma_2] \in \pi_1(X, x_0)$

$$\gamma_1 * \gamma_2 (t) = \begin{cases} \gamma_1(2t) & 0 \le t \le 1/2 \\ \gamma_2(2t-1) & 1/2 \le t \le 1 \end{cases}$$

$$(f \circ \gamma_1) * (f \circ \gamma_2) = \begin{cases} f \circ \gamma_1(2t) & 0 \le t \le 1/2 \\ f \circ \gamma_2(2t-1) & 1/2 \le t \le 1 \end{cases}$$

so $f \circ (\gamma_1 * \gamma_2) = (f \circ \gamma_1) * (f \circ \gamma_2)$

i.e. $f_*([\gamma_1] * [\gamma_2]) = f_*([\gamma_1]) * f_*([\gamma_2])$

**exercise:**

1) $(f \circ g)_* = f_* \circ g_*$

2) if $f: X \to Y$ is homotopic to $g: X \to Y$ relative to $x_0 \in X$ then $f_* = g_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$

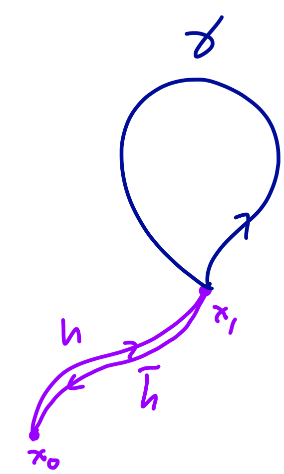
How does $\pi_1$ depend on the base point?

let $h: [0,1] \to X$ be a path with $h(0) = x_0$ and $h(1) = x_1$

if $\gamma$ is a loop in $X$ based at $x_1$, then note

$$h * \gamma * \bar{h}(t) = \begin{cases} h(3t) & 0 \le t \le 1/3 \\ \gamma(3t-1) & 1/3 \le t \le 2/3 \\ \bar{h}(3t-1) & 2/3 \le t \le 1 \end{cases}$$

is a loop based at $x_0$




**lemma 3:**

h induces an isomorphism

$$\phi_h: \pi_1(X, x_1) \to \pi_1(X, x_0)$$
$$[\gamma] \longmapsto [h * \gamma * \bar{h}]$$

## Remarks:

1) so isomorphism type of $\pi_1(X,x_0)$ only depends on path component of $X$ in which $x_0$ lies
2) The isomorphism depends on $h$ !

**Proof:** $\phi_h$ is a well-defined homomorphism (exercise)

Claim $\phi_{\bar{h}}$ is the inverse of $\phi_h$

indeed given $[\gamma] \in \pi_1(X, x_0)$

$$\phi_h \circ \phi_{\bar{h}}([\gamma]) = [\underbrace{h * \bar{h}}_{\text{loop based at } x_0} * \gamma * \underbrace{h * \bar{h}}]$$

$$= [h * \bar{h}] * [\gamma] * [h * \bar{h}]$$

but $h * \bar{h} \sim e$ as a loop based at $x_0$

so $\phi_h \circ \phi_{\bar{h}}([\gamma]) = [e] * [\gamma] * [e] = [\gamma]$

you can similarly check $\phi_{\bar{h}} \circ \phi_h = id_{\pi_1(X, x_1)}$

**Th$^m$ 4:**

$\boxed{\text{If } f: X \to Y \text{ is a homotopy equivalence, then } f_*: \pi_1(X,x_0) \to \pi_1(Y, f(x_0)) \text{ is an isomorphism}}$

to prove this we need

**lemma 5:**

Suppose $f_0, f_1 : X \to Y$ are homotopic via the homotopy

$$H: X \times [0,1] \to Y$$

let $x_0 \in X$ and $h: [0,1] \to Y : t \longmapsto H(x_0, t)$

Then

$$\begin{tikzcd} & \pi_1(Y, f_0(x_0)) \\ \pi_1(X, x_0) \arrow[ur, "(f_0)_*"] \arrow[dr, "(f_1)_*"'] \arrow[r, phantom, "\circ"] & {} \\ & \pi_1(Y, f_1(x_0)) \arrow[uu, "\phi_h"'] \end{tikzcd}$$

## Proof of Th$^{\underline{m}}$ 4:

let $g$ be the homotopy inverse of $f$ so

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, g(f(x_0)))$$

now $g_* \circ f_* \sim id_X$ so by lemma $\exists$ path $h$ s.t.

$$g_* \circ f_* = \phi_h \quad \text{an isomorphism}$$

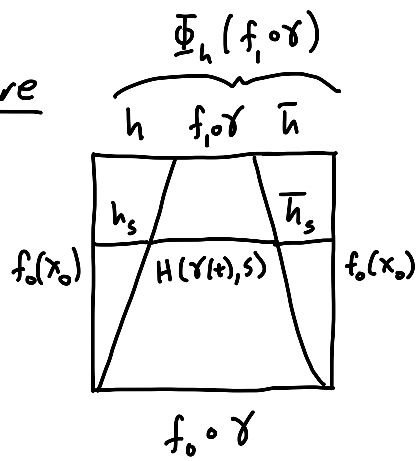
so $f_*$ is injective

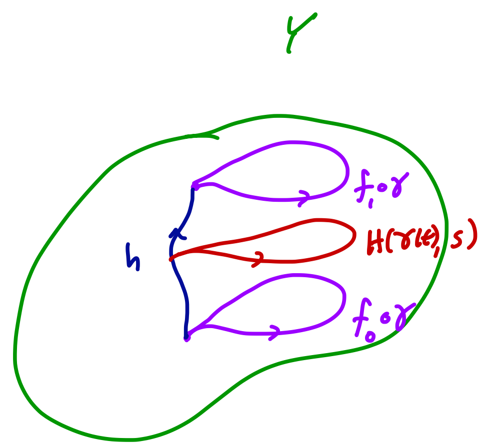
similarly $f_* \circ g_*$ is an isomorphism so $f_*$ is surjective

$\therefore$ $f_*$ an isomorphism $\square$

## Proof of lemma 5:

Picture




exercise: write out explicit homotopy $\square$

## Recap: We have a "functor"

$$\left\{ \begin{matrix} \text{pointed topological spaces,} \\ \text{pointed maps} \end{matrix} \right\} \Rightarrow \left\{ \begin{matrix} \text{groups,} \\ \text{homomorphisms} \end{matrix} \right\}$$

homotopic spaces map to isomorphic groups

homotopic functions map to the "same" homomorphism

## B. Simple Computations

**lemma 6:**

> If $X$ is contractible then $\pi_1(X,x_0)=\{1\}$ $\forall x_0 \in X$

**Proof:** If $X$ is a one point space, then $\exists!$ loop $\gamma:[0,1]\to X$ (constant loop)
so $\pi_1(X,x_0)=\{1\}$

$X$ contractible $\Rightarrow$ $X \simeq$ one point space
so done by Th$^{\underline{m}}$ 4 ∎

**Remark:** A space $X$ is called simply connected
if 1) $X$ is path connected, and
2) $\pi_1(X,x_0)=\{1\}$

so contractible spaces are simply connected

simply connected means "path connected in a particulary simple way"

**lemma 7:**

> $X$ is simply connected $\iff$ every two points in $X$ are connected by a unique homotopy class of paths in $X$

**Proof:** ($\Leftarrow$) path connected by existence of path
any loop based at $x_0$ homotopic to constant loop by uniqueness of homotopy class of path ✓

($\Rightarrow$) path connected gives existence of path $a$ to $b$
given 2 paths $\gamma,\delta:[0,1]\to X$ from $a$ to $b$
simple connectivity implies $a * \bar{b} \sim e_a$ ↙ constant $a$ path

now $\gamma \sim \gamma * (\bar{\delta} * \delta) \sim (\gamma * \bar{\delta}) * \delta \sim e_a * \delta \sim \delta$

from proof of lemma 1 even though $\delta$ a path $\bar{\delta} * \delta \sim e_b$

from proof of lemma 1

(all homotopies rell end pts of path)

**Th$^{\underline{m}}$ 8:**

$$\boxed{\pi_1(S^n) = \{1\} \ \forall n \geq 2}$$

need lemma

**lemma 9:**

let $X = A \cup B$

$A, B,$ and $A \cap B$ open and path connected

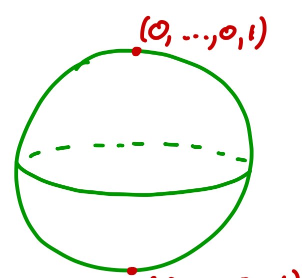
$x_0 \in A \cap B$

Then any loop $\gamma : [0,1] \to X$ based at $x_0$

can be written as $\gamma \sim \gamma_1 * \ldots * \gamma_n$

where each $\gamma_i$ is a loop in $A$ or $B$ based at $x_0$

**Proof of Th$^{\underline{m}}$ 8:**




let $A = S^n - \{(0,\ldots,0,1)\} \cong \mathbb{R}^n$

$B = S^n - \{(0,\ldots,0,-1)\} \cong \mathbb{R}^n$

$A \cap B = S^n - \left\{ \begin{matrix} (0,\ldots,-1) \\ (0,\ldots,1) \end{matrix} \right\} \cong S^{n-1} \times \mathbb{R}$

all are path connected

take $x_0 \in A \cap B$

any $[\gamma] \in \pi_1(S^n, x_0)$ can be written as

$[\gamma] = [\gamma_1][\gamma_2] \ldots [\gamma_n]$ where $[\gamma_i] \in \pi_1(A, x_0)$ or $\pi_1(B, x_0)$

by lemma 9

but $\pi_1(A, x_0) = \{1\} = \pi_1(B, x_0)$ so $[\gamma] = [e_{x_0}]$

and hence $\pi_1(S^n, x_0) = \{1\}$ $\boxplus$

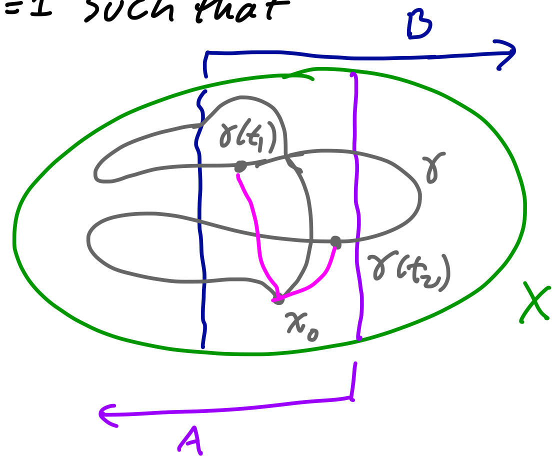
## Proof of lemma 9:

given $\gamma : [0,1] \to X$ a loop based at $x_0$

Claim: there exist $0 = t_0 < t_1 < \dots < t_n = 1$ such that

$\operatorname{im} \gamma|_{[t_{i-1}, t_i]} \subset A$ or $B$

and $\gamma(t_i) \in A \cap B$




given this let $\delta_i : [0,1] \to A \cap B$ connect $x_0$ to $\gamma(t_i)$

and $\gamma_i = \gamma|_{[t_{i-1}, t_i]}$

note: $\gamma \sim \gamma_1 * \gamma_2 * \dots * \gamma_n \sim (\gamma_1 * \bar{\delta}_1) * (\delta_1 * \gamma_2 * \bar{\delta}_2) * \delta_2 \dots (\delta_{n-1} * \gamma_n)$

loop in A or B   loop in A or B   ...   loop in A or B

Pf of Claim:

need Topology Fact (Lebesgue number lemma)

Hatcher's proof implicitly uses the Axiom of choice (when choosings intervals containing each point). The proof here is more "direct"

for a proof see any topology or analysis book

> $X$ a compact metric space
> $\{U_\alpha\}_{\alpha \in A}$ an open cover
> $\exists$ a number $\delta > 0$ (Lebesgue #)
> s.t. $\forall$ sets $S$ with $\operatorname{diam}(S) < \delta$
> $\exists \alpha$ s.t. $S \subset U_\alpha$

now $U_1 = \gamma^{-1}(A)$, $U_2 = \gamma^{-1}(B)$ is an open cover of $[0,1]$ so $\exists \delta > 0$ s.t. if $|b - a| < \delta$ then $[a,b] \subset U_i$ $i = 0$ or $1$

let $n$ be s.t. $\frac{1}{n} < \delta$

now $\gamma|_{[\frac{i}{n}, \frac{i+1}{n}]} \subset A$ or $B$

so start with $t_i = \frac{i}{n}$ $\quad i = 0, \dots n$

now if $\gamma_{[t_{i-1}, t_i]}, \gamma_{[t_i, t_{i+1}]}$ both in $A$ or $B$

the throw out $t_i$

continuing gives desired partition $\boxplus$

**Th$^{\underline{m}}$ 10:**

$$\boxed{\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)}$$

**Proof:** $\Phi : \pi_1(X, x_0) \times \pi_1(Y, y_0) \to \pi_1(X \times Y, (x_0, y_0))$

$([\gamma], [\delta]) \longmapsto [\gamma \times \delta]$ $\quad$ where $(\gamma \times \delta)(t) = (\gamma(t), \delta(t))$

is an isomorphism

exercise: 1) show $\Phi$ is well-defined homomorphism

2) show $\Phi$ is bijection (use projection) $\boxplus$

## C. Fundamental Group of $S^1$

**Th$^{\underline{m}}$ 11:**

$\pi_1(S^1, (1,0)) \cong \mathbb{Z}$

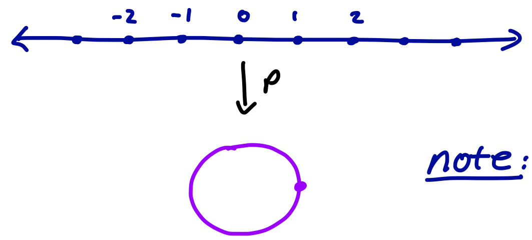
the isomorphism sends $n \in \mathbb{Z}$ to

$\gamma_n : [0,1] \to S^1 : t \mapsto (\cos 2n\pi t, \sin 2n\pi t)$

**Remark:** Proof is an example of very important technique that we will see again!

The proof involves studying the map

$$p : \mathbb{R} \to S^1 : t \mapsto (\cos 2\pi t, \sin 2\pi t)$$




<u>note</u>: $p^{-1}((1,0)) = \mathbb{Z}$

$p$ is a special case of a <u>covering map</u> (we will study these quite a bit later)

If $\gamma : [0,1] \to S^1$ is a path based at $(1,0)$ then a <u>lift of $\gamma$ based at $n \in \mathbb{Z}$</u> is a map $\tilde{\gamma} : [0,1] \to \mathbb{R}$ s.t.

1) $\tilde{\gamma}(0) = n$
2) $p \circ \tilde{\gamma}(x) = \gamma(x) \;\; \forall x$

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & S^1 \end{tikzcd}$$

<u>lemma 12:</u>

path lifting → a) for each $n \in \mathbb{Z}$, each loop $\gamma : [0,1] \to S^1$ based at $(1,0)$ lifts to a unique path $\tilde{\gamma}_n$ based at $n$.

homotopy lifting → b) if $\gamma \sim \gamma'$ are loops in $S^1$ based at $(1,0)$ and $\tilde{\gamma}_n$ and $\tilde{\gamma}'_n$ are their lifts based at $n$, then $\tilde{\gamma}_n \sim \tilde{\gamma}'_n$ rel $\{0,1\}$

<u>Proof of Th$^m$ 11 given lemma 12:</u>

Given $\gamma \in [\gamma] \in \pi_1(S^1, (1,0))$

lemma 12 a) says $\exists!\ \tilde{\gamma}_0 : [0,1] \to \mathbb{R}$

since $\tilde{\gamma}_0(1) \in p^{-1}((1,0)) = \mathbb{Z}$ we can define

$$\Phi : \pi_1(S^1, (1,0)) \to \mathbb{Z}$$
$$[\gamma] \mapsto \tilde{\gamma}_0(1)$$

lemma 12 b) say $\Phi$ is well-defined

<u>$\Phi$ surjeitive</u>: let $\tilde{\delta}^n(t) = nt$ for $t \in [0,1]$

and $\delta^n(t) = p \circ \tilde{\delta}^n$

clearly $\tilde{\delta}^n$ is a lift based at 0 of the loop $\delta^n$

and $\Phi([\delta_n]) = n$ ✓

**$\Phi$ is injective:**

suppose $\gamma, \gamma'$ are two loops in $S^1$ s.t. $\tilde{\gamma}_0(1) = \tilde{\gamma}_0'(1)$

set $\tilde{H}(s,t) = (1-t)\tilde{\gamma}_0(s) + t\,\tilde{\gamma}_0'(s)$

and $H(s,t) = p \circ \tilde{H}(s,t)$

note: $H(s,0) = \gamma(s)$

$H(s,1) = \gamma'(s)$

$H(0,t) = (1,0) = H(1,t) \quad \forall t$

i.e. $H$ is a homotopy of based loops

i.e. $\gamma \sim \gamma'$ ✓

**$\Phi$ a homomorphism:**

given $[\gamma], [\gamma'] \in \pi_1(S^1, (1,0))$

let $\tilde{\gamma}_0, \tilde{\gamma}_0'$ be the lifts of $\gamma, \gamma'$ (based at 0)

$\Phi([\gamma]) = \tilde{\gamma}_0(1) = n \qquad \Phi([\gamma']) = \tilde{\gamma}_0'(1) = m$

note: 1) $\tilde{\gamma}_n'(t) = n + \tilde{\gamma}_0'(t)$ since rt. hand side is a lift and lift is unique

2) $\tilde{\gamma}_0 * \tilde{\gamma}_n'$ is a lift of $\gamma * \gamma'$ based at 0

so $\Phi([\gamma][\gamma']) = \widetilde{\gamma * \gamma'}(1) = \tilde{\gamma}_0 * \tilde{\gamma}_n'(1) = n+m$

$= \Phi([\gamma]) + \Phi([\gamma'])$ ∎

**Proof of lemma 12:**

exercise: think about uniqueness (we will do more about this for general covering space)

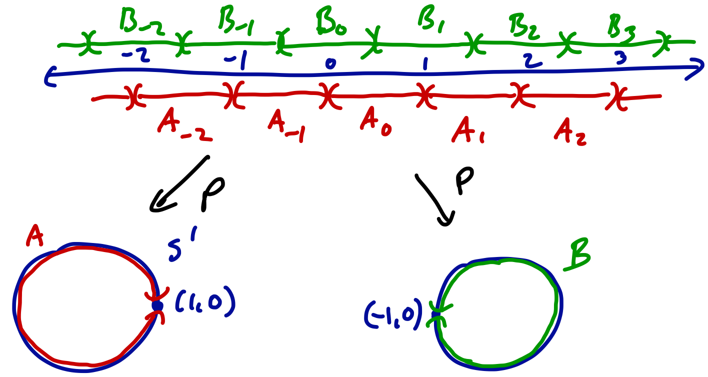
part a): let $A = S^1 - \{(1,0)\}$

$$p^{-1}(A) = \bigcup_{q \in \mathbb{Z}} \underbrace{(q, q+1)}_{A_q}$$

note: $p|_{A_q} : A_q \to A$ a homeomorphism!

similarly if $B = S^1 - \{(-1,0)\}$

then $p^{-1}(B) = \bigcup_{i\in\mathbb{Z}} \underbrace{(i-1/2,\, i+1/2)}_{B_i}$




and $p|_{B_i}: B_i \to B$ a homeomorphism

**Obvious but important observation:**

If $f: X \to S^1$ has image in $A$

then after choosing $n \in \mathbb{Z}$ $\exists$ a unique map

$$\tilde{f}: X \to A_n \subset \mathbb{R}$$

such that $p \circ \tilde{f} = f$

i.e. just set $\tilde{f} = (p|_{A_n})^{-1} \circ f$

similarly for $f(X) \subset B$.

now given a loop $\gamma: [0,1] \to S^1$ based at $(1,0)$

<u>note:</u> $\{\gamma^{-1}(A), \gamma^{-1}(B)\}$ an open cover of compact $[0,1]$

so $\exists$ Lebesgue number $\delta > 0$ for cover

choose $n$ st. $\frac{1}{n} < \delta$

note: if $t_i = \frac{i}{n}$ then $\gamma([t_i, t_{i+1}]) \subset A$ or $B$

if $\gamma([t_{i-1}, t_i])$ and $\gamma([t_i, t_{i+1}])$ lie in same $A$ or $B$ then discard $t_i$ (do this inductively on $i$)

so we get a partition $t_0 = 0 < t_1 < \dots < t_k = 1$ of $[0,1]$

st. $\gamma([t_i, t_{i+1}]) \subset \begin{cases} B & \text{for } i \text{ even} \\ A & \text{for } i \text{ odd} \end{cases}$ (since $\mathbb{Z} \cap A = \emptyset$)

now set $\tilde{\gamma}_n = (p|_{B_n})^{-1} \circ \gamma$ on $[t_0, t_1]$

note: $\tilde{\gamma}_n(t_1) \in A_k$ some $k$

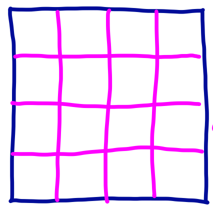
so set $\tilde{\gamma}_n = (p|_{A_k})^{-1} \circ \gamma_n$ on $[t_1, t_2]$

inductively continue to get $\tilde{\gamma}_n$ defined on $[0,1]$

since all lifts agree at endpoints $\tilde{\gamma}_n$ is continuous and clearly the desired lift! ✓

part b):

just like proof path lifting: Given homotopy $H: [0,1]\times[0,1] \to S^1$ get Lebesgue number $\delta > 0$ for $\{H^{-1}(A), H^{-1}(B)\}$ pick $n$ st. $\frac{\sqrt{2}}{n} < \delta$ then consider




each square can be lifted

exercise: write out details

Many corollaries of this computation, eg

**Cor 13:** $\boxed{\text{There is no retraction } D^2 \to \partial D^2}$

**Proof:** If there were a retraction $r: D^2 \to S^1 = \partial D^2$ then consider the inclusion map $i: S^1 \to D^2$ (as $\partial D^2$)

note $r \circ i: S^1 \to S^1$ is the identity map!

so $r_* \circ i_* = (r \circ i)_* = (id_{S^1})_* = id_{\pi_1(S^1,(1,0))}: \mathbb{Z} \to \mathbb{Z} \Rightarrow i_*$ injective

but $\pi_1(D^2,(1,0)) = \{1\}$, so $i_* =$ constant map $\cancel{\text{X}}$ injectivity

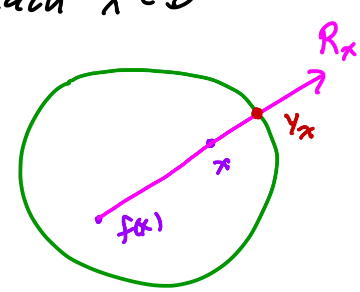
**Cor 14:** $\boxed{\text{any map } f: D^2 \to D^2 \text{ has a fixed point}}$

**Proof:** if $f: D^2 \to D^2$ had no fixed points, then for each $x \in D$

let $R_x$ = ray starting at $f(x)$ going through $x$

note: $R_x \cap \partial D^2$ in exactly 1 point $y_x$

set $g(x) = y_x$



exercise: $g(x)$ continuous (eq$^n$ for $R_x$ continuous in $x$
$\therefore$ eq$^n$ for $R_x \cap S^1$ continuous in $x$)

cleary $g$ a retraction! $\cancel{\exists}$ Cor 13 ∎

Many other applications!

1) Fundamental Th$^m$ of Algebra
2) Borsuk-Ulam Th$^m$ (abt maps $S^2 \to S^1$ and $S^2 \to \mathbb{R}^2$)
3) Ham sandwich th$^m$

⋮

see Hatcher's Book and supplement class webpage